Long Division of Polynomials

Recall the usual long division for numbers:

$$
\begin{array}{r}
7 \,\big)\, \overline{1001}\,\big(\, 143 \\
-\underline{7001} \\
301 \\
-\underline{280} \\
21 \\
-\underline{21} \\
\times
\end{array}
$$

divisor

dividend

quotient

$$
\begin{array}{l}
x-1 \,\big)\, \overline{x^2+4x-5}\,\big(\, x+5 \\
\quad \overset{(-)}{} x^2 \;\; - x \\
\qquad\quad\;\; (+) \\
\hline
\qquad\quad 5x - 5 \\
\qquad\quad 5x - 5 \\
\qquad\quad (-) \;\; (+) \\
\hline
\qquad\qquad 0
\end{array}
$$

Change signs →

Change signs →

Example 1

$$
\begin{array}{l}
2x+1 \overline{\smash{)}\, 2x^3 - 9x^2 + 7x + 6 \,(}\ x^2 - 5x + 6 \\
 \underline{\overset{(-)}{}2x^3 \overset{+}{\underset{(-)}{}} x^2} \\
 -10x^2 + 7x \\
 \underline{\underset{(+)}{-10x^2} \underset{(+)}{-\,11x}} \\
 18x \\
 \underline{\underset{(-)}{}18x} \\
 0
\end{array}
$$

Ques Why do we divide polynomials?

Ans. Our goal is to find the zeros of polynomials. For quadratics and linear polynomials this is easy because we have the quadratic formula and factoring techniques in our arsenal. But for higher degree polynomials we have to factor.

So in example 1. say that we wanted to find the zeros of $2x^3 - 9x^2 + 7x + 6$. and we were given the information that $2x+1$ is a factor. Then $2x^3 - 9x^2 + 7x + 6 = (2x+1) \cdot ?$

To find the ? we need to divide $2x^3 - 9x^2 + 7x + 6$ by $2x+1$.

Exercise 1 Divide $2x^3 + x^2 - 4x - 3$ by $x-1$.

Note

$$8 \overline{)\,610\,}(76$$
$$\underline{560}$$
$$50$$
$$\underline{48}$$
$$2$$

This means $610 = 8 \cdot 76 + 2$

Dividend Divisor Quotient Remainder

We have $\dfrac{610}{8} = 76 + \dfrac{2}{8}$

We have the following theorem for integers

Theorem 1 (Division algorithm). Let a, b be integers with $a \neq 0$. Then there exist unique integers q, r such that

$$b = a \cdot q + r$$

and $0 \leq r < a$. r is called the remainder and q is called the quotient. If $r = 0$ then a divides b.

Theorem 2 (Division algorithm for polynomials). Let $A(x)$ and $B(x)$ be polynomials with $A(x) \neq 0$ and degree of $B(x)$ is greater than or equal to the degree of $A(x)$. Then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$B(x) = A(x)\, Q(x) + R(x)$$

and degree of $R(x)$ < degree of $A(x)$.

Exercise 2 See if you can prove Thm 1 and 2.

$$x^2+1 \,\overline{\big)\, 3x^4 + 2x^3 + x^2 + 4 \,}\big(\, 3x^2 + 2x - 2$$

$$\begin{array}{rrrr}
3x^4 & & +3x^2 & \\
(-) & & (-) & \\
\hline
 & 2x^3 & -2x^2 & & +4 \\
 & 2x^3 & & +2x & \\
 & (-) & & (-) & \\
\hline
 & & -2x^2 & -2x & +4 \\
 & & -2x^2 & & -2 \\
 & & (+) & & (+) \\
\hline
 & & & -2x & +6
\end{array}$$

$$\therefore \frac{3x^4+2x^3+x^2+4}{x^2+1} = 3x^2+2x-2 + \frac{-2x+6}{x^2+1}$$

Exercise Divide $2x^5+3x^2+12$ by x^3-3x-4

Divide $8x^4 - 5x^3 + 7x - 2$ by $2x^2+1$

$$2x^2+1 \,\big)\, 8x^4 - 5x^3 + 7x - 2 \,\big(\, 4x^2 - \frac{5}{2}x - 2$$

$$8x^4 + 4x^2$$

$$(-) \quad (-)$$

$$-5x^3 - 4x^2 + 7x - 2$$

$$-5x^3 \qquad -\frac{5}{2}x$$

$$(+) \quad (+)$$

$$-4x^2 + \frac{19}{2}x - 2$$

$$-4x^2 \qquad -2$$

$$(+) \quad (+)$$

$$\frac{19}{2}x$$

$\therefore$

$$\frac{8x^4 - 5x^3 + 7x - 2}{2x^2+1} = 4x^2 - \frac{5}{2}x - 2 + \frac{\frac{19}{2}x}{2x^2+1}.$$

Synthetic Division

When the divisor is $x-a$ then we ______ can use synthetic division.

Example Divide $x^4 - x^3 - 2x + 2$ by $x+1$

$x+1 = x-(-1)$.

$$
\begin{array}{r|rrrrr}
-1 & 1 & -1 & 0 & -2 & 2 \\
 & \downarrow & -1 & 2 & -2 & 4 \\
\hline
 & 1 & -2 & 2 & -4 & 6
\end{array}
$$

6: Remainder

$$x^3 - 2x^2 + 2x - 4$$

$$\therefore \frac{x^4 - x^3 - 2x + 2}{x+1} = x^3 - 2x^2 + 2x - 4 + \frac{6}{x+1}$$

Example Divide $3x^5 - 2x^3 + x^2 - 7$ by $x+2$

$x+2 = x-(-2)$

$$
\begin{array}{r|rrrrrr}
-2 & 3 & 0 & -2 & 1 & 0 & -7 \\
 & \downarrow & -6 & 12 & -20 & 38 & -76 \\
\hline
 & 3 & -6 & 10 & -19 & 38 & -83
\end{array}
$$

$$\therefore \frac{3x^5 - 2x^3 + x^2 - 7}{x+2} = 3x^4 - 6x^3 + 10x^2 - 19x + 38 - \frac{83}{x+2}$$

Exercise Divide $2x^3 - x + 3$ by $x-1$.

Exercise Divide $x^3 - x^2 - 9x + 9$ by $x-1$.

$$x-1 \overline{) 2x^3 + x^2 - 4x - 3} \left(2x^2 + 3x + 7 \right.$$

$$2x^3 - 2x^2$$

$$(+) \quad (+)$$

$$3x^2 - 4x - 3$$

$$3x^2 - 3x$$

$$(-) \quad (+)$$

$$7x - 3$$

$$7x - 7$$

$$(+) \quad (+)$$

$$4$$

$$3x^2 - 5 \overline{) 8x^4 - 5x^3 + x - 6} \left(\frac{8}{3}x^2 - \frac{5}{3}x + \frac{40}{9} \right.$$

$$8x^4 - \frac{40}{3}x^2$$

$$(-) \quad (+)$$

$$-5x^3 + \frac{40}{3}x^2 + x - 6$$

$$-5x^3 \quad \frac{25}{3}x$$

$$(+) \quad (-)$$

$$\frac{40}{3}x^2 - \frac{22}{3}x - 6$$

$$\frac{40}{3}x^2 \quad -\frac{200}{9}$$

$$(+) \quad (+)$$

$$-\frac{22}{3}x + \frac{146}{9}$$

$3 \cdot \frac{40}{9}$

$1 - \frac{25}{3}$

$\frac{3-25}{3}$

$-\frac{22}{3}$

-200

$\frac{200}{9} - 6 \quad 146$

$\frac{200 - 54}{9}$

$x^4 + 9x^3 - 4x + 9$ by $x - 1$

$$
\begin{array}{r|rrrrr}
1 & 1 & 9 & 0 & -4 & 9 \\
 & & 1 & 10 & 10 & 6 \\
\hline
 & 1 & 10 & 10 & 6 & \boxed{15}
\end{array}
$$

$x^3 + 10x^2 + 10x + 6$ Remaindr 15

$x^3 - x^2 + 7$ by $x - 3$

$$
\begin{array}{r|rrrr}
3 & 1 & -1 & 0 & 7 \\
 & & 3 & 6 & 18 \\
\hline
 & 1 & 2 & 6 & \boxed{25}
\end{array}
$$